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An Isoperimetric Inequality and the First Steklov Eigenvalue

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Let (M^n, g) be a compact Riemannian manifold with boundary. In this paper we give upper and lower estimates for the first nonzero Steklov eigenvalue

$$\Delta\varphi = 0 \qquad \text{in } M,$$

where ν_1 is a positive real number. The estimate from below is for a star-shaped domain on a manifold whose Ricci curvature is bounded from below. The upper estimate is for a convex manifold with nonnegative Ricci curvature and is given in terms of the first nonzero eigenvalue for the Laplacian on the boundary. We prove a comparison theorem for simply connected domains in a simply connected manifold. We exhibit annuli domains for which the comparison theorem fails to be true. In (J. F. Escobar, *J. Funct. Anal.* **60** (1997), 544–556) we introduced the isoperimetric constant $I(M)$ defined as

$$I(M) = \inf_{\Omega \subset M} \frac{\operatorname{Vol}(\Sigma)}{\min\{\operatorname{Vol}(\Omega_1), \operatorname{Vol}(\Omega_2)\}},$$

where $\Omega_1 = \Omega \cap \partial M$ is a nonempty domain with boundary in the manifold ∂M, $\Omega_2 = \partial M - \Omega_1$, and $\Sigma = \partial\Omega \cap \operatorname{int}(M)$, where $\operatorname{int}(M)$ is the interior of M. We proved a Cheeger's type inequality for ν_1 using the constant $I(M)$. In this paper we give upper and lower estimates for the constant I in terms of isoperimetric constants of the boundary of M. © 1999 Academic Press

Let (M^n, g) be a compact Riemannian manifold with boundary. In our previous paper [E] we studied the Steklov eigenvalue problem:

$$\begin{aligned} \Delta\varphi &= 0 && \text{in } M, \\ \frac{\partial\varphi}{\partial\eta} &= \nu\varphi && \text{on } \partial M, \end{aligned} \tag{1}$$

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where ν is a real number. We gave estimates from below for the first non-zero eigenvalue and introduced the isoperimetric constant $I(M)$ defined as

$$I(M) = \inf_{\Omega \subset M} \frac{\mathrm{Vol}(\Sigma)}{\min\{\mathrm{Vol}(\Omega_1), \mathrm{Vol}(\Omega_2)\}}, \tag{2}$$

where $\Omega_1 = \Omega \cap \partial M$ is a non-empty domain with boundary in the manifold ∂M, $\Omega_2 = \partial M - \Omega_1$, and the set $\Sigma = \partial \Omega \cap \mathrm{int}(M)$ where $\mathrm{int}(M)$ is the interior of M. We established a relationship between $I(M)$ and ν_1, the first non-zero eigenvalue of problem (1)—see Theorem 10 in [E].

In this paper we give upper and lower estimates for the constant I, an upper estimate for the eigenvalue ν_1, and prove some comparison theorems for the first non-zero Steklov eigenvalue of a two-dimensional manifold.

Let Ω be a bounded domain in the plane. We want to estimate the isoperimetric constant $I(\Omega)$. In this case it takes the form

$$I(\Omega) = \inf_{x,\, y \in \partial\Omega} \frac{d_\Omega(x, y)}{d_{\partial\Omega}(x, y)}, \tag{3}$$

where d_X represents the distance function on the set X.

In our first proposition we evaluate the constant I for the Euclidean ball in the plane.

Proposition 1. *Let $B^2 \subset R^2$ be the Euclidean ball. Then $I(B^2) = 2/\pi$.*

Proof. Without lost of generality we can assume that B^2 is the unit ball in R^2. Using the symmetries of the ball it is enough to consider the quotient (3) when we fixed a point $x \in S^1$. Let $x = (1, 0)$. We represent a point in $\partial B^2 = S^1$ by $\theta \in [0, 2\pi)$ with the convention that the point $(1, 0)$ is represented by 0. By the symmetries of the ball it is enough to restrict ourselves to the points $0 \leqslant \theta \leqslant \pi$. Thus

$$I(B^2) = \inf_\theta \frac{[2(1-\cos\theta)]^{1/2}}{\theta} \qquad \theta \in [0, \pi].$$

To show that the function $f(\theta) = [1-\cos\theta]^{1/2}/\theta$ is decreasing in the interval $(0, \pi)$ it is enough to show that $f'(\theta) < 0$ on the interval $(0, \pi)$. That is equivalent to the assertion that for $\theta \in (0, \pi)$ the inequality $\theta \sin\theta < 2(1-\cos\theta)$ holds. This is easily verified to be true using differentiation. Therefore the minimum for the function $f(\theta)$ is achieved when θ is π, that is, the antipodal point on S^1, and hence the result.

Theorem 2. *Let $\Omega \subset R^2$ be a bounded domain with rectifiable boundary. Then $I(\Omega) \leqslant 2/\pi$. Equality holds only for the ball.*

Proof. Let L denote the perimeter of $\partial\Omega$. First observe that

$$I(\Omega)=\inf_{x,\,y\in\partial\Omega}\frac{d_\Omega(x,y)}{d_{\partial\Omega}(x,y)}\leqslant \inf_{\substack{x,\,y\in\partial\Omega\\ d_\Omega(x,y)=L/2}}\frac{d_\Omega(x,y)}{d_{\partial\Omega}(x,y)}$$

$$=\frac{2}{L}\inf_{\substack{x,\,y\in\partial\Omega\\ d_\Omega(x,y)=L/2}} d_\Omega(x,y).$$

After stretching the domain Ω we can assume that the infimum in the right-hand side of the last inequality is 1. Then a theorem due to Falconer in [F] asserts that $L\geqslant\pi$ and that $L=\pi$ if the curve bounds a convex set of constant width. Therefore $I(\Omega)\leqslant 2/\pi$. Since the only curve of constant width each of whose diametral chords bisects the perimeter is the circle [HS], it follows that equality holds only for the ball.

Remark 3. After we introduced the isoperimetric constant I in our paper [E], Huisken in [H] showed that a very similar constant to $I(\Omega)$, where $\partial\Omega$ is smooth, in non-decreasing under the curve shortening flow, and he used this fact to give a shorter proof of Grayson's heat flow theorem in [G]. If one uses Grayson's theorem and then observes that I is non-decreasing (the same calculation as that used in [H]) one proves Theorem 1 for smooth domains. However, with this approach one obtains a weaker result because in Theorem 2 we only assume that $\partial\Omega$ is rectifiable.

Remark 4. In the class of manifolds we clearly have that $I(M)\leqslant 1$. There are several manifolds that satisfy $I(M)=1$, for example, any geodesic ball, B_r, of radius r, in the unit sphere, $S^2\subset R^3$, with $r\geqslant\pi/2$.

Next we evaluate the constant I for the Euclidean ball in any dimension. In the process we prove an interesting isoperimetric equality for some minimal surfaces in the ball (Eq. (4) below). This is done in Theorem 5 below. Upon showing our proof to A. Freire, he pointed out to the author that for the case when $n\geqslant 3$ Theorem 5 was previously proved by Bokowski and Sperner [BS]. Their proof uses a non-standard symmetrization argument. Ours is shorter and the method can be extended to obtain estimates on manifolds. In fact, one can give an upper estimate and a lower estimate of the constant $I(M)$ in terms of an isoperimetric constant of ∂M. Estimates for convex domains in R^n were obtained by Bokowski and Sperner in [BS]. Ours is for non-convex domains and our method is different (Theorem 6 below).

THEOREM 5. *Let B^n be the Euclidean ball in R^n. Then $I(B^n)=2\omega_{n-1}/\sigma_{n-1}$ where ω_{n-1} and σ_{n-1} represent the volume of the unit ball and the unit sphere in R^{n-1} respectively.*

Proof. In view of Theorem 2 we assume that $n \geqslant 3$. It is clear that we can restrict ourselves to the case where Σ is a minimal hypersurface on the ball. Standard arguments in geometric measure theory imply that there is a minimizer.

The minimizer hits the boundary of the ball in a constant contact angle because the first variation of the area A of Σ is

$$\left.\frac{dA}{dt}\right|_{t=0} = -\int_{\Sigma} \langle H, E\rangle + \int_{\partial\Sigma} \langle E, \eta_{\Sigma}\rangle,$$

where E is the variational vector field, H is the mean curvature, and η_{Σ} is the outward normal vector of $\partial\Sigma$ in Σ. The variation of the area B of the domain Ω_1 (we assume that $\mathrm{Area}(\Omega_1) \leqslant \mathrm{Area}(\Omega_2)$) is

$$\left.\frac{dB}{dt}\right|_{t=0} = \int_{\partial\Omega} \langle E, \eta_1\rangle,$$

where η_1 is the outward normal vector of $\partial\Omega_1$ in Ω_1. Thus a critical point of the functional I satisfies that the mean curvature vector H of Σ is zeero and that $\int_{\partial\Sigma} \langle E, (\eta_{\Sigma} - \lambda\eta_1\rangle = 0$ where λ is a Lagrange multiplier. Since the last integral identity holds for all variations E that are tangent to the boundary of the ball we conclude that η_{Σ} makes a constant angle θ with the normal vector to the boundary of the ball. Observe that in the case when $\mathrm{Area}(\Omega_1) = \mathrm{Area}(\Omega_2)$, Σ is minimal and orthogonal to the boundary. In any case we observe that

$$(n-1)\,\mathrm{Vol}_{n-1}(\Sigma) = c\,\mathrm{Vol}_{n-2}(\partial\Sigma), \tag{4}$$

where $c = \cos(\theta)$. In order to see this consider the function $f(x) = \frac{1}{2}r^2(x)$ where $r(x)$ is the distance to the origin. Let t be the distance to the hypersurface Σ. Since Σ is minimal and the Hessian of the function f is the identity matrix we find that

$$n = \Delta_{R^n} f = \frac{\partial^2 f}{\partial t^2} + \Delta_{\Sigma} f = 1 + \Delta_{\Sigma} f.$$

Therefore

$$(n-1)\,\mathrm{Vol}(\Sigma) = \int_{\Sigma} \Delta_{\Sigma} f = \int_{\partial\Sigma} r\,\frac{\partial r}{\partial\eta_{\Sigma}},$$

where the last equality is obtained by integrating by parts.

Using the fact that $r(x) = 1$ and $\partial r/\partial\eta_{\Sigma} = c$ on $\partial\Sigma$, because Σ makes a constant angle with ∂B we find that the equality (4) holds. Writing

$$\frac{\mathrm{Vol}(\Sigma)}{\min\{\mathrm{Vol}(\Omega_1), \mathrm{Vol}(\Omega_2)\}} = \frac{\mathrm{Vol}(\Sigma)}{\mathrm{Vol}(\partial\Sigma)}\,\frac{\mathrm{Vol}(\partial\Omega_1)}{\min\{\mathrm{Vol}(\Omega_1), \mathrm{Vol}(\Omega_2)\}},$$

and using the equality (4) we get that

$$I(B^n) = \frac{c}{n-1} \; \frac{\mathrm{Vol}(\partial \Omega_1)}{\min\{\mathrm{Vol}(\Omega_1),\ \mathrm{Vol}(\Omega_2)\}}.$$

Using the isoperimetric inequality on the sphere it is easy to check that $\Omega_1 = B_r$, a geodesic ball on the sphere S^{n-1}, and hence Σ is the intersection of a hyperplane in R^n with B^n whose boundary is equal to the boundary of B_r. From this we can compute c and now it is straightforward to verify that actually $\Omega_1 = B_{\pi/2}$, $c = 1$, and $I(B^n) = 2/(n-1)(\sigma_{n-2}/\sigma_{n-1})$. Since $\sigma_{n-2} = (n-1)\,\omega_{n-1}$ this completes the proof of our theorem.

The previous calculation can be generalized to manifolds as follows. Let (M^n, g) be a compact Riemannian manifold with boundary. Assume that the Ricci tensor satisfies $\mathrm{Ric}(g) \geqslant -(n-1)\,kg$. Let Ω be a minimizer for the isoperimetric constant $I(M)$. Assume that $\partial\Omega \cap M = \Sigma$ is star-shaped with respect to some point $x_0 \in M$ (not necessarily in Σ). That is to say, there exists a positive constant c_0 such that $\nabla r \cdot \eta_\Sigma \geqslant c_0$ where $r(x) = d(x, x_0)$. Let $f(x) = \frac{1}{2} r(x)^2$. Let t denotre the geodesic distance to Σ. The minimality of Σ implies that on Σ

$$\Delta_M f = \frac{\partial^2 f}{\partial t^2} + \Delta_\Sigma.$$

The Laplacian comparison theorem implies that

$$\Delta_M f = 1 + r\Delta_M r \leqslant 1 + (n-1)\coth(\sqrt{k}r)\sqrt{k} \leqslant 1 + (n-1)(1 + \sqrt{k}r).$$

Thus

$$\Delta_M f \leqslant n + \sqrt{k}\, r_{\max} = c_1$$

where $r_{\max} = \max_{x \in \Sigma} r(x)$.

Let Hf denotes the Hessian of the function f on M. Then $Hf(\partial_t, \partial_t) = \partial^2 f/\partial t^2$. Assume that $Hr \geqslant 0$ on Σ; then $Hf = dr \otimes dr + rHr \geqslant 0$ on Σ. Therefore $\Delta_\Sigma \leqslant c_1$. Integrating the last inequality over Σ and using integration by parts we get

$$c_1 \,\mathrm{Vol}(\Sigma) \geqslant \int_\Sigma \Delta_\Sigma f = \int_{\partial\Sigma} r\, \frac{\partial r}{\partial \eta_\Sigma} \geqslant c_0 r_{\min} \,\mathrm{Vol}(\partial\Sigma),$$

where $r_{\min} = \min_{x \in \partial\Sigma} r(x)$. Since

$$\begin{aligned} I(M) &= \inf_{\Omega \subset M} \frac{\mathrm{Vol}(\Sigma)}{\min\{\mathrm{Vol}(\Omega_1),\ \mathrm{Vol}(\Omega_2)\}} \\ &= \inf_{\Omega \subset M} \frac{\mathrm{Vol}(\Sigma)}{\mathrm{Vol}(\partial\Sigma)} \; \frac{\mathrm{Vol}(\partial\Sigma)}{\min\{\mathrm{Vol}(\Omega_1),\ \mathrm{Vol}(\Omega_2)\}} \end{aligned}$$

we get that

$$I(M) \geqslant \frac{r_0 c_0}{c_1} \min_{i=1, \ldots, k} I_1(N_i), \tag{5}$$

where $\partial M = N_1 \cup \cdots \cup N_k$, N_i is a connected manifold, and

$$I_\beta(\partial M) = \inf_{\Omega \subset \partial M} \frac{(\mathrm{Vol}(\partial \Omega))^\beta}{\min\{\mathrm{Vol}(\Omega_1), \mathrm{Vol}(\Omega_2)\}}.$$

A way to get upper estimates for the constant $I(M)$ when M is an arbitrary manifold is the following. We write

$$\begin{aligned} I(M) &= \inf_{\Omega \subset M} \frac{\mathrm{Vol}(\Sigma)}{\min\{\mathrm{Vol}(\Omega_1), \mathrm{Vol}(\Omega_2)\}} \\ &= \inf_{\Omega \subset M} \frac{\mathrm{Vol}(\Sigma)}{(\mathrm{Vol}(\partial \Sigma))^\alpha} \frac{(\mathrm{Vol}(\partial \Sigma))^\alpha}{\min\{\mathrm{Vol}(\Omega_1), \mathrm{Vol}(\Omega_2)\}} \end{aligned}$$

for any positive number α. If M is a bounded domain in R^n then we can use Almgren's isoperimetric inequality (see [A]) which states that

$$\mathrm{Vol}(\Sigma) \leqslant \gamma(n-1)(\mathrm{Vol}(\partial \Sigma))^{(n-1)/(n-2)}$$

where the constant $\gamma(n-1)$ is the optimal in the classical isoperimetric inequality in R^{n-1}. Therefore for a domain M in R^n we obtain

$$I(M) \leqslant \gamma(n-1) \max_{i=1, \ldots, k} I_{(n-1)/(n-2)}(N_i).$$

The same argument can be used with any α for which we know

$$\mathrm{Vol}(\Sigma) \leqslant C(n)(\mathrm{Vol}(\partial \Sigma))^\beta$$

and we have an estimate of the constant $I_\beta(\partial M)$. When $\beta = (n-1)/(n-2)$ Hoffman and Spruck [HS] gave an estimate of the constant $C(n)$ for general manifolds. However, the only known case where the constant is sharp is in R^n, proven by Almgren in [A].

Then we have proved the following theorem:

Theorem 6. *Let (M^n, g) be an n-dimensional compact manifold with boundary, with $n \geqslant 3$. Let $\partial M = N_1 \cup \cdots \cup N_k$ where N_i is a connected manifold. Then there exists a positive constant $c(n, g)$ such that*

$$I(M) \leqslant c(n, g) \max_{i=1, \ldots, k} I_{(n-1)/(n-2)}(N_i).$$

If M is a bounded domain in R^n then $c(n, g) = \gamma(n-1)$ the best constant in the classical isoperimetric inequality in R^{n-1}. Assume that the Ricci tensor satisfies that $\mathrm{Ric}(g) \geqslant -(n-1)\,kg$ *where $k \geqslant 0$ and $I(M)$ has a minimizer Ω, with $\partial\Omega \cap M$ star-shaped with respect to a point $x_0 \in M$. If the distance function to the point x_0 is convex on Σ then inequality* (5) *holds.*

We remark that Yau in [Y] gave lower estimates for $I_\beta(\partial M)$ when $\beta = 1$, $(n-1)/(n-2)$, and later Croke in [C] improved Yau's estimates. Thus combining Croke's estimates with the estimates of Theorem 6 we get estimates from below for the isoperimetric constant $I(M)$. The convexity of the distance function to the point x_0 can be easily checked if the sectional curvature of M is non-positive or more generally if x_0 is a pole. If the sectional curvature is less than or equal to a positive constant K, the Hessian Comparison Theorem implies that when $r_{\max} \leqslant \pi/2\sqrt{K}$ the distance function to the point x_0 is convex on Σ.

Let M^2 be a simply connected domain in the plane. Weinstock in [Wk] gave the sharp estimate $\nu_1 \leqslant 2\pi/L$ where L is the length of the boundary. Later Hersch, Payne, and Schiffer in [HPS] gave an elegant and simpler proof of Weinstock's Theorem and proved the sharp estimate $\nu_1 \nu_2 \leqslant 4\pi^2/L^2$ where ν_2 is the second non-zero Steklov eigenvalue. In [HPS] upper estimates for all eigenvalues of the Steklov problem in multiple connected domains in the planes are given. We want to remark that all theorems in [HPS] generalize to two-dimensional compact manifolds with boundary in a straightforward way. Using this generalization we can prove the following theorem.

THEOREM 7. *Let (M^2, g) be a simply connected complete Riemannian manifold with constant Gaussian curvature. Let $\Omega \subset M$ be a bounded, simply connected domain with* $\mathrm{Area}(\Omega) = \mathrm{Area}(B_r(x_0))$ *where $B_r(x_0) \subset M$ is a geodesic ball of radius r with center $x_0 \in M$. Then*

$$\nu_1(\Omega) \leqslant \nu_1(B_r(x_0)).$$

Equality holds only when Ω is isometric to $B_r(x_0)$.

Proof. Multiplying the metric on M by a positive constant and using Remark 3 in [E] we can assume that the Gaussian curvature of M is 1, 0, or -1. Therefore M is isometric to the unit sphere, the Euclidean plane, or the hyperbolic space of curvature -1. Wienstock's Theorem says that $\nu_1(\Omega) \leqslant 2\pi/L$ where L is the perimeter of $\partial\Omega$. The isomerimetric inequality on M implies that $L \geqslant L(r)$ where $L(r)$ represents the perimeter of $\partial B_r(x_0)$; equality holds only when Ω is isometric to $B_r(x_0)$. Therefore

$$v_1(\Omega) \leqslant \frac{2\pi}{L(r)}.$$

In [E, Example 5], we showed that if $B_r(x_0) \subset S^2$ then

$$v_1(B_r(x_0)) = \cot(r) + \tan\left(\frac{r}{2}\right).$$

Observe that

$$\cot(r) + \tan\left(\frac{r}{2}\right) = \frac{\cos^2((r/2) - \sin^2(r/2)}{2\sin(r/2)\cos(r/2)} + \frac{\sin(r/2)}{\cos(r/2)}$$
$$= \frac{\cos^2(r/2) + \sin^2(r/2)}{2\sin(r/2)\cos(r/2)} = \frac{1}{\sin(r)}.$$

In [E, Example 6], we showed that the if $\mathbb{H}$ is the hyperbolic space with constant curvature -1, and $B_r(x_0) \subset \mathbb{H}$, then

$$v_1(B_r(x_0)) = \coth(r) - \tanh\left(\frac{r}{2}\right).$$

Calculation similar to that performed previously shows that

$$\coth(r) + \tanh\left(\frac{r}{2}\right) = \frac{1}{\sinh(r)}.$$

For the Euclidean ball it is well known that $v_1(B_r(x_0)) = 1/r$.

Thus, in all cases, $v_1(B_r(x_0)) = 2\pi/L(r)$ and hence we have the theorem.

We can generalize the last theorem to non-positive curved spaces in the following way:

THEOREM 8. *Let (M^2, g) be a complete simply connected Riemannian manifold with non-positive Gaussian curvature. Let Ω be a bounded, simply connected domain in M^2 with* $\text{Area}(\Omega) = \text{Area}(B_r(0))$ *where $B_r(0)$ is the Euclidean ball of radius r with center at* 0. *Then*

$$v_1(\Omega) \leqslant v_1(B_r(0)).$$

Equality holds only when Ω is isometric to the Euclidean ball $B_r(0)$.

Proof. Wienstock's Theorem says that $\nu_1(\Omega) \leqslant 2\pi/L$ where L is the perimeter of $\partial\Omega$. Weyl's isoperimetric inequality on non-positive curvature manifolds [Wl] says that

$$L^2 \geqslant 4\pi A$$

where $A = \text{Area}(\Omega)$ and the equality holds only for the Euclidean ball. Therefore we get that $L \geqslant 2\pi r$, which implies that $\nu_1(\Omega) \leqslant 1/r = \nu_1(B_r(0))$.

In what follows we show that there are non-simply-connected domains (annuli) for which Theorem 6 and Theorem 7 do not hold.

Consider the annulus $\Omega = A_{r_1, r_2} = \{(r, \theta) \mid r_1 < r < r_2,\ \theta \in S^1\}$ endowed with a rotationally invariant metric $ds^2 = dr^2 + f^2(r)\, d\theta^2$.

To calculate the first non-zero eigenvalue for the Steklov problem in Ω we use the separation of variables method and observe that the space $L^2(\Omega) = L^2(r_1, r_2) \otimes L^2(S^1)$.

Let $e_{n,1} = \cos(n\theta)/\sqrt{\pi}$, $e_{n,2} = \sin(n\theta)/\sqrt{\pi}$ for $n = 1, 2, \ldots$, and let $e_0 = 1/\sqrt{2\pi}$ and $a_{n,i}(r)$ be the functions satisfying

$$\begin{gathered}\frac{1}{f(r)} \frac{d}{dr}\left(f(r) \frac{d}{dr} a_{n,i}(r)\right) - \frac{n a_{n,i}(r)}{f(r)^2} = 0 \qquad \text{in} \quad (r_1, r_2) \\ a_{n,i}(r_1) = -\nu_{n,i} a_{n,i}(r_1) \qquad a_{n,i}(r_2) = \nu_{n,i} a_{n,i}(r_2),\end{gathered} \tag{6}$$

where $n = 0, 1, 2, \ldots$. The set $u_{n,i}(r, \theta) = a_{n,i}(r)\, e_{n,i}(\theta)$ forms an orthogonal basis for $L^2(\Omega)$.

From the variational characterization of the eigenvalues we have

$$\begin{aligned}\nu_{n,i} &= \frac{\int_\Omega |\nabla u_{n,i}|^2 f\, dr\, d\theta}{\int_{\partial\Omega} u_{n,i}^2 f\, d\theta} \\ &= \frac{\int_{r_1}^{r_2} ((d/dr)\, a_{n,i}(r))^2 f\, dr + n \int_{r_1}^{r_2} (a_{n,i})^2 f^{-1}\, dr}{a_{n,i}(r_1)^2 f(r_1) + a_{n,i}(r_2)^2 f(r_2)}.\end{aligned}$$

Therefore one has either $\nu_1 = \alpha$ or $\nu_1 = \beta$, where

$$\alpha = \inf_{a \in C^\infty} \frac{\int_{r_1}^{r_2} ((d/dr)\, a(r))^2 f\, dr + \int_{r_1}^{r_2} a^2 f^{-1}\, dr}{a(r_1)^2 f(r_1) + a(r_2)^2 f(r_2)} \tag{7}$$

and

$$\beta = \inf_{a \in \mathscr{C}} \frac{\int_{r_1}^{r_2} ((d/dr)\, a(r))^2 f\, dr}{a(r_1)^2 f(r_1) + a(r_2)^2 f(r_2)},$$

where $\mathscr{C} = \{a \in C^\infty(r_1, r_2) \mid a(r_1)\, f(r_1) + a(r_2)\, f(r_2) = 0\}$.

It is easy to check that

$$\beta = \frac{f(r_1) + f(r_2)}{f(r_1)\, f(r_2) \int_{r_1}^{r_2} (ds/f(s))}.$$

In order to calculate α observe that a minimizer always exists. The minimizer satisfies the Euler–Lagrange equation of the variational problem (7), which is Eq. (6) with $n=1$. Let $a(r)$ be such a solution. Using the change of coordinates $t = t(r) = \int^r (ds/f(s))$ we take the equation (6) to the cylinder. In fact, if we let $b(t) = a(r)$ then $b(t)$ satisfies the ordinary differential equation $d^2 b(t)/dt^2 - b(t) = 0$, which has the general solution $c_1 e^t + c_2 e^{-t}$ where c_i is a constant, $i = 1, 2$. Therefore the general solution of Eq. (6) with $n=1$ is $c_1 e^{t(r)} + c_2 e^{-t(r)}$. Without loss of generality we can assume that $c_1 = 1$. Using the boundary condition one calculates that α is equal to

$$\frac{\left[\begin{array}{l}(f(r_1)+f(r_2))(e^{2t_0}+1) \\ \qquad -\sqrt{(f(r_1)+f(r_2))^2\,(e^{2t_0}+1)^2 - 4f(r_1)\, f(r_2)(e^{2t_0}-1)^2}\end{array}\right]}{2f(r_1)\, f(r_2)(e^{2t_0}-1)}$$

where $t_0 = t(r_2) - t(r_1)$. The condition $\text{Area}(\Omega) = \text{Area}(B_{r_0})$ implies that

$$\int_0^{r_0} f(s)\, ds = \int_{r_1}^{r_2} f(s)\, ds. \tag{8}$$

Now we restrict our discussion to the case of the sphere, the plane, and the hyperbolic space. In these cases $f(r) = \sin(r)$, r, and $\sinh(r)$, respectively. The function t is equal to $\ln(\tan(r/2))$, $\ln(r)$, and $\ln(\tanh(r/2))$, respectively.

In the following examples we have used Maple to calculate $\min\{\alpha, \beta\}$, which is the first non-zero eigenvalue on the annulus, and to compare it with $v_1(B_{r_0})$. In the case of the sphere, let $\Omega_1 = A_{r_1, r_2} \subset S^2$ with $r_1 = \pi/6$ and $r_2 = 5\pi/6$; then one finds that

$$v_1(\Omega_1) = \beta = \frac{4}{\ln((2+\sqrt{3})/(2-\sqrt{3}))}.$$

A computation shows that for r_0 defined by Eq. (8) we have

$$v_1(B_{r_0}) = \frac{1}{\sqrt{1-(1-\sqrt{3})^2}} < v_1(\Omega_1).$$

In the next example we have that the first non-zero eigenvalue for an annulus is achieved by α instead of β and yet it is bigger than the one on the geodesic ball with the same volume. This is the case for $\Omega_2 = A_{r_1, r_2} \subset S^2$ where $r_1 = \pi/5$ and $r_2 = 4\pi/5$. We have

$$\nu_1(\Omega_2) = \alpha > \nu_1(B_{r_0}) = \frac{1}{\sqrt{1 - (\frac{1}{2} - \frac{1}{2}\sqrt{5})^2}},$$

where r_0 satisfies Eq. (8).

In the case of the plane we have that if $\Omega_3 = A_{r_1, r_2}$ with $r_1 = \frac{1}{12}$ and $r_2 = 10^4$, then

$$\nu_1(\Omega_3) = \alpha = \frac{14400000001 - \sqrt{207353088143999520001}}{2399980000}$$

and

$$\nu_1(B_{r_0}) = \frac{2\sqrt{6805293}}{52173913} < \nu_1(\Omega_3).$$

In the hyperbolic space $\mathbb{H}^2$ we have the following example. Let $\Omega_4 = A_{r_1, r_2} \subset \mathbb{H}^2$, with $r_1 = \frac{1}{5000}$ and $r_2 = 2$. Then $\nu_1(\Omega_4) = \alpha$ and it is equal to

$$\frac{B(A + \sqrt{A^2 - 4\sinh(1/5(10)^3)\sinh(2)((\tanh^2(1) - \tanh^2(10^{-4}))^2)/B^2})}{2\sinh(1/5(10)^3)\sinh(2)[\tanh^2(1) - \tanh^2(10^{-4})]}$$

where $A = \sinh(1/5(10)^3) + \sinh(2)$ and $B = (\tanh^2(1) + \tanh^2(10^{-4}))$.

A straightforward calculation shows that

$$\nu_1(B_{r_0}) = \frac{1}{\sqrt{(1 + \cosh(1/5000) + \cosh(2))^2 - 1}}$$

where r_0 satisfies Eq. (8).

In this case we also have $\nu_1(B_{r_0}) < \nu_1(\Omega_4)$.

In the next proposition we give an upper estimate for the first non-zero Steklov eigenvalue for higher-dimensional manifolds.

THEOREM 9. *Let M^n be a compact Riemannian manifold with boundary, with $n \geqslant 3$. Assume that the Ricci curvature of M is non-negative and the second fundamental form of ∂M, $\pi \geqslant 0$. Then*

$$(\min_{x \in \partial M} h(x))\nu_1 < \frac{2\lambda_1}{n-1},$$

where λ_1 is the first non-zero eigenvalue of the Laplacian on ∂M.

Proof. A particular case of Reilly's formula (see [R, p. 46]) says that, for a smooth function f defined on M, the following identity holds if $u=\partial f/\partial\eta$ on ∂M:

$$\int_M (\Delta f)^2-|\mathrm{Hess}\, f|^2=\int_M \mathrm{Ric}(\nabla f, \nabla f)+\int_{\partial M} (\bar{\Delta} f+(n-1)\, h_g u)\, u$$
$$-\int_{\partial M} (\bar{\nabla} f, \bar{\nabla} u)+\int_{\partial M} \pi(\bar{\nabla} f, \bar{\nabla} f).$$

Here $\bar{\Delta}$, $\bar{\nabla}$ respectively represent the Laplacian and the gradient on ∂M with respect to the induced metric on ∂M. Let φ_1 be the first eigenfunction of the Laplacian on ∂M. Let f be the harmonic function on M satisfying that $f=\varphi_1$ on ∂M. Applying Reilly's formula we find that

$$\int_{\partial M} (\bar{\Delta}\varphi_1+(n-1)\, h_g u)\, u-\int_{\partial M} (\bar{\nabla}\varphi_1, \bar{\nabla} u)<0.$$

Therefore

$$-2\lambda_1 \int_{\partial M} \varphi_1 u=2\int_{\partial M} \bar{\Delta}\varphi_1 u< -(n-1)\int_{\partial M} h_g u^2.$$

Thus

$$\frac{2\lambda_1}{(n-1)}>\frac{\int_{\partial M} hu^2}{\int_{\partial M} \varphi_1 u}\geqslant \big(\min_{x\in\partial M} h(x)\big)\, \frac{\int_{\partial M} u^2}{\int_{\partial M} \varphi_1 u}. \tag{9}$$

Recall that the first non-zero Steklov eigenvalue has the following characterization:

$$\nu_1=\min_{\int_{\partial M} f=0} \frac{\int_M |\nabla f|^2}{\int_{\partial M} f^2}.$$

It is easy to show that

$$\min_{\int_{\partial M} f=0} \frac{\int_M |\nabla f|^2}{\int_{\partial M} f^2}=\min_{\Delta f=0} \frac{\int_{\partial M} u^2}{\int_{\partial M} fu}.$$

As a consequence of the last two equalities and the inequality (9) we get our estimate and this completes the proof of our theorem.

Now we discuss an estimate that holds in any star-shaped domain on a manifold where the Ricci curvature is bounded from below. A similar estimate in the case of star-shaped domains in the Euclidean case has been obtained by Bramble and Payne in [BP].

Let $\Omega \in M$ be a star-shaped domain with respect to a point $x_0 \in M$ and let $B(x_0, r_0) \subset \Omega$ be a geodesic ball with center x_0 and radius r_0. Consider a function $u : \bar{\Omega} \to \mathbb{R}$ with the property

$$\int_{\partial B_{r_0}(p)} u = 0. \tag{10}$$

The divergence theorem states that for a vector field X defined on $\bar{\Omega}$,

$$\int_{\Omega - B_{r_0}(p)} \operatorname{div}(Xu^2) = \int_{\partial\Omega} u^2(X, \eta) - \int_{\partial B_{r_0}(p)} u^2(X, \eta)$$

where η is the outward normal vector.

If $(X, \eta) \geqslant C_1$ on $\partial\Omega$ and $(X, \eta) \leqslant C_2$ on $\partial B_{r_0}(p)$ we find that

$$C_1 \int_{\partial\Omega} u^2 \leqslant C_2 \int_{\partial B_{r_0}(p)} u^2 + \int_{\Omega - B_{r_0}(p)} (\operatorname{div} X) u^2 + 2 \int_{\Omega - B_{r_0}(p)} u(X, \nabla u).$$

Because u satisfies condition (10) we can assert that

$$\int_{B_{r_0}} |\nabla u|^2 \geqslant v_1(B_{r_0}) \int_{\partial B_{r_0}} u^2$$

where $v_1(B_{r_0})$ represents the first non-zero Steklov eigenvalue for the set B_{r_0}. Using this and Cauchy's inequality with a positive function g we obtain

$$C_1 \int_{\partial\Omega} u^2 \leqslant C_2 v_1^{-1}(B_{r_0}) \int_{B_{r_0}} |\nabla u|^2 + \int_{\Omega - B_{r_0}} (\operatorname{div} X + g^{-1} |X|^2) u^2$$
$$+ \int_{\Omega - B_{r_0}} g\, |\nabla u|^2.$$

Assuming

$$\operatorname{div} X + g^{-1} |X|^2 \leqslant 0, \tag{11}$$

we find that

$$C_1 \int_{\partial\Omega} u^2 \leqslant C_3 \int_{\Omega} |\nabla u|^2$$

where $C_3 = \max\{C_2 v_1^{-1}(B_{r_0}), \sup_{x \in \Omega - B_{r_0}} g(x)\}$.

In order to apply the previous discussion to the first eigenfunction for the Steklov problem, φ_1, we consider the function $u=\varphi_1+a$ where $a=-(1/\mathrm{Vol}(\partial B_{r_0}))\int_{\partial B_{r_0}}\varphi_1$. Then u satisfies the equality (10). Therefore

$$\int_\Omega |\nabla\varphi_1|^2=\int_\Omega |\nabla u|^2 \geqslant \frac{C_1}{C_3}\int_{\partial\Omega} u^2 \geqslant \frac{C_1}{C_3}\int_{\partial\Omega}\varphi_1^2.$$

Hence $\nu_1(\Omega)\geqslant C_1/C_3$.

We apply the previous discussion to the following theorem.

THEOREM 11. *Let (M^n, g) be an n-dimensional Riemannian manifold and $\Omega\subset M$ be a star-shaped domain with respect to the point x_0. Let $B_{r_0}(x_0)\subset\Omega$ and assume that the Ricci tensor*, $\mathrm{Ric}(g)$, *satisfies that* $\mathrm{Ric}(g)\geqslant -(n-1)\,kg$ *on Ω where $k\geqslant 0$. Let $r(x)=\mathrm{dist}(x, x_0)$. Then*

$$\nu_1(\Omega)\geqslant \frac{C_0}{f^n(r_{\max})\,C_3}$$

where $C_0=\min_{x\in\partial\Omega}(\nabla r(x), \eta(x))$, $f(r)=(1/\sqrt{k})\ \sinh(\sqrt{k}r)$, $r_{\max}=\max_{x\in\partial\Omega} r(x)$, *and* $C_3=\max\{f'(r_{\max})\,f^{n-1}(r_{\max}),\ \nu_1^{-1}(B_{r_0})\,f(r_0)^{-n}\}$.

Proof. We exhibit a vector field X satisfying the inequality (11). Let $X=\nabla r/f(r)^n$ and choose $g(r)=f'f^{n-1}$. Then we have

$$\mathrm{div}\,X=\frac{\Delta r}{f(r)^n}-nf^{-n-1}f'.$$

Since $\mathrm{Ric}(g)\geqslant -(n-1)\,kg$ the Laplacian comparison theorem implies that

$$\mathrm{div}\,X\leqslant (n-1)\ \frac{f'}{f^{n+1}}-n\ \frac{f'}{f^{n+1}}=\ -\frac{f'}{f^{n+1}}.$$

Hence

$$\mathrm{div}\,X+g^{-1}\,|X|^2\leqslant\ -\frac{f'}{f^{n+1}}+f'f^{n-1}\ \frac{1}{f^{2n}}=0.$$

Now observe that on ∂B_{r_0} we have $(X, \eta)=1/f(r_0)^n$ and for $x\in\partial\Omega$,

$$(X(x), \eta(x))\geqslant (\min_{x\in\partial\Omega}\ f^{-n}(x))\langle\nabla r(x), \eta(x)\rangle\geqslant f^{-n}(r_{\max})\,C_0,$$

where $C_0 > 0$ because the domain Ω is star-shaped with respect to the point x_0. Clearly

$$\sup_{x \in \Omega - B_{r_0}} g(x) \leqslant f'(r_{\max}) \, f^{n-1}(r_{\max})$$

because the functions f and f' are increasing.

By combining the above estimates with our previous discussion we get the theorem.

Remark 12. In the case in which $k = 0$ we can use in place of the discussion f in the proof of the last theorem the function $f(r) = r$.

Finally, we close this paper with the following conjecture.

CONJECTURE. *Let (M^n, g) be a compact Riemannian manifold with boundary and dimension $n \geqslant 3$. Assume that* $\mathrm{Ric}(g) \geqslant 0$ *and that the second fundamental form π satisfies $\pi \geqslant k_0 I$ on ∂M, $k_0 > 0$. Then*

$$\nu_1 \geqslant k_0 .$$

Equality holds only for the Euclidean ball of radius k_0^{-1}.

In our previous paper [E] we obtained the estimate $\nu_1 > k_0/2$ (when $n \geqslant 3$). The analogous conjecture when $n = 2$ was proved for domains in the Euclidean space by Payne [P] and for manifolds with non-negative Gaussian curvature by the author (see Theorem 1 in [E]).

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